Experiments on grid-excited solitons in a positive-ion–negative-ion plasma

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(Received 16 June 1992)

Experiments on the grid excitation and subsequent propagation of solitons in a positive-ion–negative-ion–electron plasma are described. In the experiments, the ratio of the negative-ion density to the positive-ion density in a quasineutral plasma is a parameter that can be varied. Hence solitons that are described by either the Korteweg–de Vries or the modified Korteweg–de Vries equations can be experimentally investigated.

PACS number(s): 52.35.Mw, 52.35.Fp

I. INTRODUCTION

The unearthing and subsequent description of the properties of solitons that propagate in a quasineutral plasma that consists of three species—positively charged ions, negatively charged ions, and electrons—is a topic of current interest. Characteristics of solitons that propagate in such a plasma differ from those that are found in a plasma that consists of only the normal two components: positively charged ions and electrons. A parameter that is important for classifying these properties is the ratio of the negative-ion density to the positive-ion density, ϵ. When small-amplitude nonlinear perturbations are described by the fluid equations and by Poisson's equation in a two-component plasma ($\epsilon=0$) with no magnetic field, it is possible to use a procedure called the reductive perturbation method to derive a Korteweg–de Vries (KdV) equation [1]. This equation admits a soliton solution corresponding to a compression of the positive ions that we classify as a compressive soliton. In a plasma containing negative ions the reductive perturbation method is applied to the fluid equations for positive and negative ions and to Poisson's equation. The resulting KdV equation admits a soliton solution that corresponds to either a compression or rarefaction of the positive ions depending on ϵ [2]. The respective solutions are known as compressive or rarefactive solitons. At a certain critical value of ϵ, which we identify as ϵ_c, the coefficient of the nonlinear term in the KdV equation vanishes and the derivation must be carried out to the next order. This leads to a modified Korteweg–de Vries (mKdV) equation [3]. The mKdV equation admits both compressive and rarefactive soliton solutions.

Laboratory experiments have been performed to investigate the properties of both KdV and mKdV solitons. In these experiments, the negative ions were created by introducing an additional gas into a normal plasma. The required property of the gas was that electrons would easily attach themselves to the atoms or molecules of this additional gas. A gas such as sulfur hexaflouride (SF_6) has this property. Waves are excited using one of three methods: (i) double plasma excitation, (ii) solid plate excitation, and (iii) grid excitation.

Nakamura and his collaborators performed a series of experiments that investigated soliton propagation in a plasma with negative ions. Using the double plasma (DP) method of wave excitation, they were able to launch both rarefactive and compressive mKdV solitons if $\epsilon=\epsilon_c$ [4] and rarefactive KdV solitons if $\epsilon>\epsilon_c$ [5]. The study of mKdV solitons included an investigation of an elastic head-on collision employing a triple plasma device. They also examined large-amplitude solitary waves described with a pseudopotential method that is applicable when the small-amplitude perturbation method is no longer valid [6]. This work was further extended and summarized in a series of papers [7].

We have recently completed a series of experiments in a large multidipole device that contained an Ar-SF_6 plasma. The goal of these experiments was to ascertain the properties of the excitation of solitons that were launched by applying a negative voltage step to a solid metal plate [8]. The magnitude of the excitation voltage was 1–4 times the electron temperature. In these experiments, the values of ϵ were greater than the critical value ϵ_c. We interpreted the excitation as being caused by the rapid explusion of the negative ions adjacent to the plate due to the polarity of the applied voltage. Therefore rarefactive KdV solitons were examined in that series of experiments. Since the velocity of a KdV soliton is amplitude dependent, the observation of an overtaking collision of different-amplitude rarefactive solitons was observed [9]. In addition, the partial reflection and the partial transmission of a soliton entering a density inhomogeneity was noted. These experiments were done in planar geometry. An experiment studying the soliton characteristics in cylindrical and spherical geometries was also performed [10]. Finally, by focusing an initially planar soliton with a concave mirror, a two-dimensional soliton could be created [11].

As a continuation and natural extension of investigating the properties of solitons in a plasma with negative ions, we have now performed a series of experiments using a third method of launching ion waves: grid excitation. In addition to the rarefactive KdV solitons that are excited when $\epsilon > \epsilon_c$, we were able to launch compressive and rarefactive mKdV solitons under the condition that $\epsilon \approx \epsilon_c$. A brief review of the KdV and the mKdV equations along with some expected soliton properties are given in Sec. II. The experimental setup is described in Sec. III. The experimental results and the interpretation are presented in Sec. IV. Section V is the conclusion.

II. KdV AND mKdV EQUATIONS

The derivations of Das and Tagare [2] and Watanabe [3] started with the nonlinear equations of continuity and motion for both the cold positive and cold negative ions. Poisson's equation was used to couple Boltzmann electrons to both ion species. The background plasma in the absence of any perturbation was quasineutral and no steady drifts were present.

Using the same reductive perturbation technique of Washimi and Taniuti [1], Das and Tagare [2] derived a KdV equation which is valid to lowest order to describe a propagating wave in a plasma with n_0 positive ions, ϵn_0 negative ions, and $(1-\epsilon)n_0$ Boltzmann electrons. It is written as

$$\frac{\partial \phi}{\partial t} + A\phi\frac{\partial \phi}{\partial x} + \frac{1}{2}\frac{\partial^3 \phi}{\partial x^3} = 0 . \tag{1}$$

In a three-component plasma, the coefficient A is given by

$$A = \frac{-1}{2} + \frac{3}{2\Xi^4(1-\epsilon)}\left[1-\frac{\epsilon}{M^2}\right] , \tag{2}$$

where $M = M_-/M_+$ is the ratio of the negative- to positive-ion mass. The parameter Ξ is the dimensionless speed of the fast ion acoustic mode [12,2] at which the wave propagates. For a three-component plasma it is given by

$$\Xi = \left[\frac{1}{1-\epsilon}\left[1+\frac{\epsilon}{M}\right]\right]^{1/2} = \frac{v}{c_s} .$$

The solution of the KdV equation can be written as

$$\phi = \phi_0 \operatorname{sech}^2\left\{\left[\frac{A\phi_0}{6}\right]^{1/2}\frac{1}{\lambda_D}\left[x - c_s \Xi t\left[1+\frac{A\phi_0}{3}\right]\right]\right\} , \tag{3}$$

with λ_D, the Debye length and c_s, the ion acoustic velocity, given in standard symbols by

$$\lambda_D = \left[\frac{\epsilon_0 T_e}{n_e e^2}\right]^{1/2} = \left[\frac{\epsilon_0 T_e}{(1-\epsilon)n_0 e^2}\right]^{1/2} ,$$

$$c_s = \left[\frac{T_e}{M_+}\right]^{1/2} .$$

Two important features of the KdV soliton that can be gleaned from the solution given in (3) are that the product of the amplitude and the square of the width is a constant and that the velocity of propagation depends upon the amplitude.

At the leading edge of a small perturbation where ϕ and $\partial\phi/\partial x$ have opposite signs, the KdV equation without the dispersive term becomes

$$\frac{\partial \phi}{\partial t} = -A\phi\frac{\partial \phi}{\partial x} = A\left|\phi\frac{\partial \phi}{\partial x}\right| .$$

This edge will grow and steepen only if A and ϕ have the same sign. The sign of A determines the polarity of the growing perturbation and therefore determines whether the soliton solution will be compressive $(+)$ or rarefactive $(-)$. When $\epsilon=0$, $A=1$ and (1) reduces to the KdV equation of a two-component plasma [1]. For $\epsilon=1$, $A=-\frac{1}{2}$. Both values of A are independent of M. There is then a critical density ϵ_c, where A vanishes. Only compressive KdV solitons are found in the plasma when $\epsilon < \epsilon_c$ and only rarefactive KdV solitons propagate when $\epsilon > \epsilon_c$.

The critical density for a three-component plasma is

$$\epsilon_c = \frac{3M^2+2M+3}{4}\left[1-\left[1-\frac{16M^2}{(3M^2+2M+3)^2}\right]^{1/2}\right] . \tag{4}$$

This depends upon the mass ratio M. When the negative ions are very light, ϵ_c diminishes to zero. When the mass ratio is large, $M \gg 1$, ϵ_c approaches $\frac{2}{3}$.

At $\epsilon=\epsilon_c$ Watanabe [3] carried the reductive perturbation expansion to the next order and derived the mKdV equation

$$\frac{\partial \phi}{\partial t} + B\phi^2\frac{\partial \phi}{\partial x} + \frac{1}{2}\frac{\partial^3 \phi}{\partial x^3} = 0 , \tag{5}$$

with the coefficient B for a three-component plasma given by

$$B = \frac{-1}{4} + \frac{15}{4\Xi^6(1-\epsilon_c)}\left[1+\frac{\epsilon_c}{M^3}\right] . \tag{6}$$

This nonlinear coefficient is positive for all values of M and ϵ_c. If the dispersive term in (5) is ignored, the mKdv equation at the leading edge of a perturbation becomes

$$\frac{\partial \phi}{\partial t} = \phi \left| B\phi \frac{\partial \phi}{\partial x} \right| .$$

Both rarefactive and compressive perturbations can steepen and form solitons in the same plasma since $\partial\phi/\partial t$ and ϕ will have the same sign.

The solution of the mKdV equation can be written as

$$\phi = \phi_0 \operatorname{sech} \left\{ \left[\frac{B\phi_0^2}{3} \right]^{1/2} \frac{1}{\lambda_D} \left[x - c_s \Xi t \left[1 + \frac{B\phi_0^2}{6} \right] \right] \right\} . \quad (7)$$

Two important features of the mKdV soliton that can be ferreted out from the solution given in (7) are that the product of the amplitude and the width is a constant and that the velocity of propagation depends upon the amplitude.

III. EXPERIMENTAL SETUP

The experiments were performed in a large multidipole plasma device that has been described elsewhere [13]. The device consists of several hollow rectangular tubes that were filled with approximately 1500 small permanent magnets. Each row had magnet pole faces pointing in the same direction and the rows alternated in magnetic-pole orientation. The tubes were arranged in a cage structure (80 cm diam$\times$110 cm length) and the entire structure was inserted in a large vacuum chamber whose base vacuum pressure was maintained beneath 10^{-6} Torr by continuous pumping. Argon was bled into the chamber using a double-valve system to a neutral pressure of $(1-2)\times10^{-4}$ Torr. Ionizing electrons emitted from joule heated filaments were accelerated by an 80-V bias supply to the cage which served as the anode. Typical plasma numbers monitored with a Langmuir probe were as follows: electron density $n_e \approx 10^8-10^9$ cm^{-3} and electron temperature $T_e \approx \frac{1}{2}-\frac{3}{2}$ eV. The ion temperature T_i measured with an energy analyzer was $T_i < T_e/10$. The plasma potentials ranged from +5 V to +1 V and decreased as the amount of SF_6 was increased. The plasma was homogeneous in the experimental region. The small amount of noise present in the plasma was due primarily to ripple in the filament bias and decreased with the addition of the SF_6.

Small amounts of sulfur hexafluoride were bled into the chamber through a separate double-valve system. The negative-ion concentration ϵ was measured by monitoring the reduction of the electron saturation current of the Langmuir probe [14]. We assume the introduction of SF_6 does not alter the positive-ion density, n_0, and define the parameter ϵ to be

$$\epsilon = \frac{n_i^-}{n_0} = \frac{n_0 - n_e}{n_0} = 1 - \frac{I_{\text{es}}}{I_{\text{es}}^{(0)}}$$

where I_{es} and $I_{\text{es}}^{(0)}$ are the Langmuir probe saturation currents with and without the SF_6 and n_i^- and n_e are the negative ion and electron densities. This will be used later is describing the experimental results.

Several species of negative ions could be present in the plasma [15]. SF_6 has a high attachment cross section for electron energies less than 1 eV yielding SF_6^- and SF_5^-. This predominates with plasma electrons where $T_e \sim 1$ eV. Bombardment of SF_6 by the 80-eV primary electrons and scattered secondaries also yields F^- through several dissociative processes. Although $SF_5{}^-$ and $SF_6{}^-$ may be involved in the soliton excitation, the lighter F^- will dominate the ion acoustic wave and therefore determine the propagation characteristics. No measurement was made of the relative abundances of these negative ions.

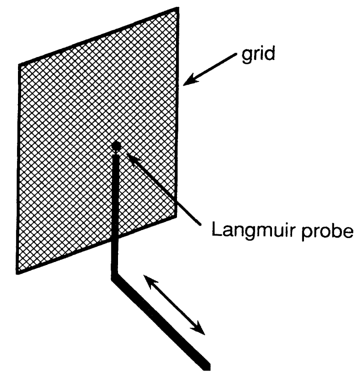

FIG. 1. The experimental setup.

Density perturbations were excited by applying a voltage signal to a fine mesh stainless-steel grid (20$\times$20 cm^2; 0.15 mm wire diameter with a transparency of approximately 45%). This is shown in Fig. 1. The amplitude $\Delta\phi$ of the applied voltage signal was $|\Delta\phi| < 4$ V and the repetition frequency of the voltage step was ≈ 50 kHz. All signals were detected with a 3-mm-diam spherical Langmuir probe biased positive with respect to the plasma potential in order to detect perturbations in the electron saturation current. The probe could be scanned in the region in front of the grid. The perturbations in current were passed through a resistor to ground and the resulting voltage perturbations were displayed on a LeCroy digital oscilloscope that was externally triggered by the signal generator. An interactive data-acquisition routine was used to send the experimental data directly from the scope to a Macintosh computer via a general purpose interface bus (GPIB) port. The amplitude perturbation of signals identified as solitons, $\delta n/n$, was less than 4%.

IV. EXPERIMENTAL RESULTS AND INTERPRETATION

In a plasma with negative ions, it has been shown above that the soliton genre is determined by the parameter ϵ. Figure 2 summarizes the soliton characteristics for different values of ϵ. In this figure, the separation distance between the launching grid and the Langmuir probe was kept constant at 8 cm. The excitation voltage is shown at the bottom except that for Fig. 2(b), which is at the bottom of Fig. 4.

Figure 2(a) depicts the response in a pure Ar plasma ($\epsilon=0$). An ion burst that evolves into a compressive KdV soliton is launched when the excitation voltage increases. We have previously interpreted this grid excitation with a model based on velocity modulation of ions

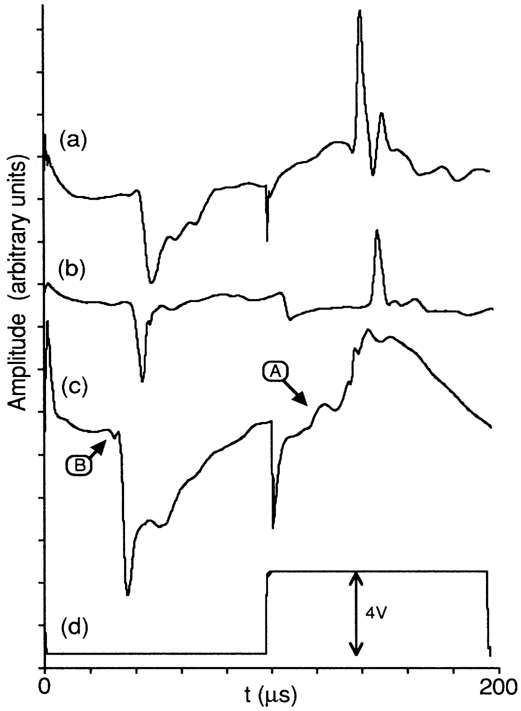

FIG. 2. Effect of changing the negative-ion concentration ϵ in the plasma. In each case the relative perturbation $\delta n/n_0$ was less than 2%. The probe was 8 cm from the grid. (a) $\epsilon=0<\epsilon_c$; (b) $\epsilon=0.07\approx\epsilon_c$; (c) $\epsilon=0.44>\epsilon_c$. The signals A and B are identified as ion bursts. (d) The excitation voltage for (a) and (c). The excitation voltage for (b) is at the bottom of Fig. 4.

that pass through the grid [13]. A density perturbation that evolves into a rarefactive dispersing Airy function is launched after the excitation voltage decreases. This response exists in a plasma with negative ions for $\epsilon<\epsilon_c$.

The signal shown in Fig. 2(b) depicts the response in an Ar plasma into which a critical amount of SF_6 has been added ($\epsilon=\epsilon_c$). In this case, a rarefactive mKdV soliton is launched after the excitation voltage decreases and a compressive mKdV soliton is launched after the excitation voltage increases. A three-component plasma consisting of Ar^+ ions, electrons, and F^- ions with $M=0.476$ theoretically spawns $\epsilon_c=0.102$. If $SF_6{}^-$ is assumed to be the negative ion, $\epsilon_c=0.54$. We found $\epsilon_c=0.07$ for the data in Fig. 2(b). This agrees closely with the former value. It is the lighter F^- ions that determine ϵ_c.

Figure 2(c) depicts the response in an Ar plasma into which a larger amount of SF_6 has been added ($\epsilon=0.44>\epsilon_c$). A rarefactive KdV soliton is launched after the excitation voltage decreases and a dispersing Airy function is launched after the excitation voltage increases. In terms of the velocity modulation model, this suggests that the negative ions are involved in the soliton excitation as was found for the plate excitation [8].

In Fig. 3 we summarize the relationship between the measured amplitude and width of the solitons represented by Fig. 2 to what is predicted in Eqs. (3) and (7). Figure 3(a) depicts the relation obtained for a compressive

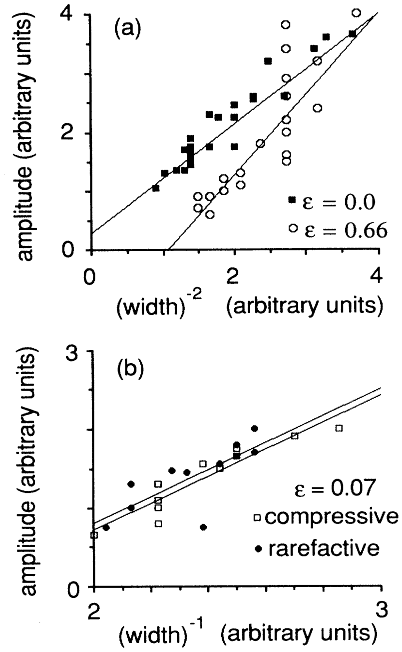

FIG. 3. Soliton amplitude-width relations. (a) KdV solitons; (b) mKdV solitons.

KdV soliton ($\epsilon=0$) and a rarefactive KdV soliton ($\epsilon=0.66>\epsilon_c$). In both cases, the amplitude of the soliton was inversely proportional to the square of the width. Figure 3(b) depicts the relation for the compressive and rarefactive mKdV solitons in Fig. 2(b) with $\epsilon=0.07\approx\epsilon_c$. Both mKdV solitons satisfied the requirement that the amplitude was inversely proportional to the width.

The propagation of the compressive and rarefactive mKdV solitons is shown in a sequence of pictures displayed in Fig. 4. Figures 2(c) and 4 show that a small foot precedes the soliton and Airy function. The feet are labeled with the letters A and B. We suggest that the feet are positive or negative ions that are accelerated through the grid and are involved in the wave excitation with the same protocol that has been described previously [8]. For a constant negative-ion concentration the foot velocities should scale with the excitation potential $\phi_{\rm ex}$. The ratio is

$$\frac{v_{\rm A}}{v_{\rm B}}\approx\frac{\left[\dfrac{2e|\phi_{\rm ex}|}{M_+}\right]^{1/2}}{\left[\dfrac{2e|\phi_{\rm ex}|}{M_-}\right]^{1/2}}=\sqrt{M}\ .$$

We found that when $\phi_{\rm ex}$ was varied from 4 to 11 V the velocity ratio of the feet shown in Fig. 2(c) was 1.8 implying that the negative-ion species observed here was $SF_5{}^-$ or $SF_6{}^-$ with $\sqrt{M}=1.8$ or 1.9, respectively. The $SF_6{}^-$ rarefactive foot could be seen for $0.4<\epsilon<0.6$ indicating that the heavier negative-ion species participate in the soliton excitation. In this range of ϵ there was no foot that we could attribute to F^-. This could have been damped or masked by the dc signal. For $\epsilon>0.6$ no rare-

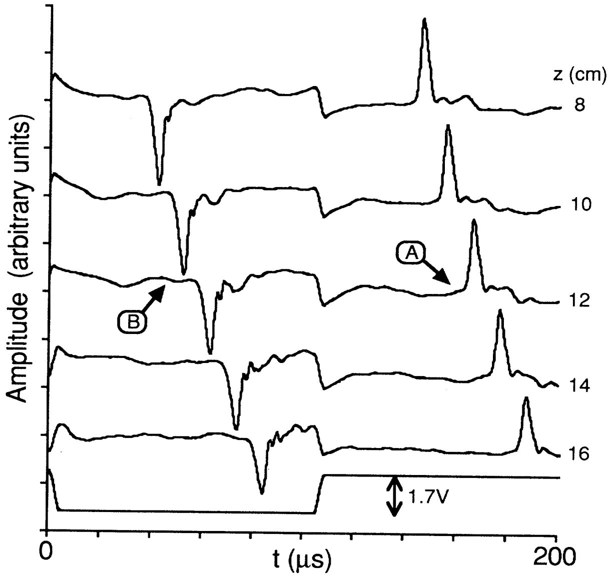

FIG. 4. Propagation of mKdV solitons ($\epsilon=0.07\approx\epsilon_c$) with $\delta n/n_0\approx 1\%$. The signals A and B are identified as ion bursts. The bottom trace is the excitation signal.

factive foot existed. Nakamura found that the velocity of the feet in Fig. 4 was nearly equal to $(M_F/M_{\rm Ar})^{1/2}$ when the two mKdV solitons had equal amplitudes [5].

Soliton velocities obtained from time-of-flight measurements when ϵ was varied are displayed in Fig. 5. The curves are the fast ion acoustic-mode velocities for a three-component cold-ion plasma consisting of Ar^+ ions, electrons, and (1) F^- or (2) SF_6^- ions. Since $\delta n/n<4\%$ for each measurement, no correction was made for the soliton Mach velocities because they varied less than 2% from the fast mode speed. The soliton speeds approach the fast mode velocity for SF_6^- when $\epsilon>0.6$, perhaps due to an increase in the relative concentration of SF_6^- to F^- as ϵ increases. This could shift the fast mode velocity from curve (1) to curve (2).

Signals detected at 8 cm from the grid that are launched with constant excitation voltages ($\phi_{\rm ex}=2.3$ V) having different rise times are shown in Fig. 6. The

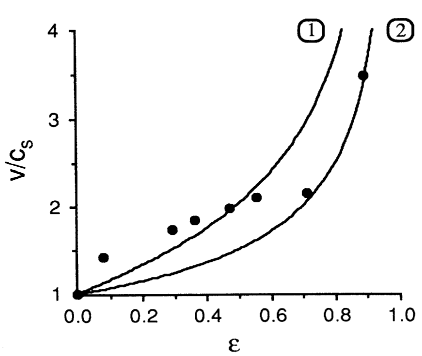

FIG. 5. Variation of soliton velocity with ϵ. The labeled curves are the three-component fast-ion acoustic speeds with a negative-ion species of (1) F^- and (2) SF_6^-.

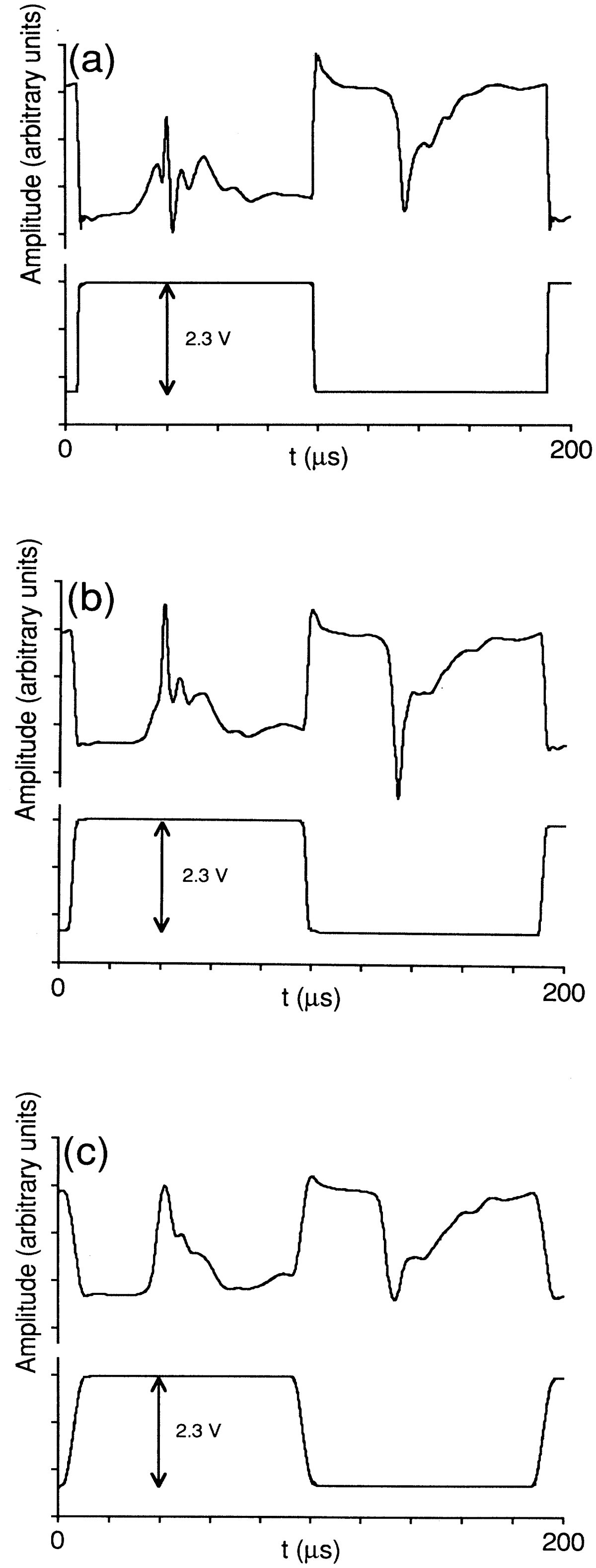

FIG. 6. Effect of rise and decay time τ on mKdV soliton excitation. The probe was 8 cm from the grid and $\delta n/n_0<2\%$. The bottom trace is the excitation signal. (a) $\tau=0.4$ μs $<\tau_{\rm opt}$; (b) $\tau=2.2$ μs $\approx\tau_{\rm opt}$; (c) $\tau=7.6$ μs $>\tau_{\rm opt}$.

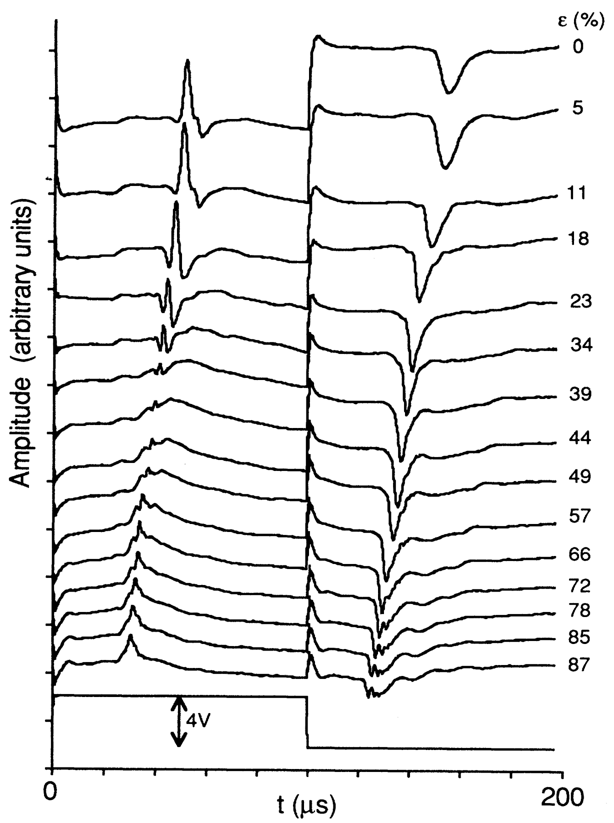

FIG. 7. Signals detected 10 cm from the grid when ϵ is varied. For each signal, $\delta n / n_0 < 2\%$. The bottom trace is the excitation signal.

negative-ion concentration was adjusted to $\epsilon = 0.10 \approx \epsilon_c$ so that mKdV solitons were excited. The formation of mKdV solitons for a fixed sinusoidal-pulse width has been shown to be amplitude dependent [5]. When the amplitude is fixed, there is an optimum rise-decay time $\tau_{\rm opt}$, which produces the largest amplitude mKdV soliton as seen in Fig. 6. We also found this for rarefactive KdV solitons where $\tau_{\rm opt}$ is a decay time that varies when ϵ is varied. The optimum excitation of solitons in a plasma with negative ions therefore depends upon $\phi_{\rm ex}$, ϵ, and $\tau_{\rm opt}$. In a two-component plasma ($\epsilon = 0$) the largest compressive KdV soliton is excited when $e\phi_{\rm ex}/\tau_{\rm opt} \approx T_e f_{pi}/\alpha$ where αf_{pi}^{-1} is some portion of the ion plasma period [13]. Our observations imply a variation of this model may be applicable for $\epsilon > 0$.

Figure 7 displays a sequence of detected signals that were observed 10 cm from the grid as the negative-ion concentration is varied in small discrete increments. The decay time of the excitation voltage was not optimized and therefore $\tau < \tau_{\rm opt}$. As ϵ changes from 0 to 0.9, several features can be discerned: (i) The fast mode velocity increases and the detected signals occur earlier in time; (ii) The rarefactive Airy function response to a negative-going pulse at $\epsilon = 0$ steepens and becomes a rarefactive soliton near $\epsilon = 0.11 \approx \epsilon_c$. The amplitude of this soliton increases as ϵ increases and then decreases when $\epsilon > 0.4$. This soliton breaks up into a number of small solitons for $\epsilon > 0.6$; and (iii) The response to a positive-going pulse observed as a compressive soliton at $\epsilon = 0$ sinks below the dc signal for $\epsilon > \epsilon_c$ and a compressive Airy function evolves. Since τ was not optimized, the signals detected at $\epsilon = 0.11 \approx \epsilon_c$ are similar to Fig. 6(a).

V. CONCLUSION

The excitation of KdV and mKdV solitons using a fine mesh grid in a plasma with negative ions has been observed. Fast ions ahead of the soliton are noted. The excitation has been interpreted in terms of velocity modulation mechanism. The characteristics of the propagating solitons are summarized.

ACKNOWLEDGMENTS

The authors acknowledge insightful discussions with Paul Hansen about wave steepening. We thank Al Scheller for constructing most of the experimental apparatus. This work was supported in part by the National Science Foundation Grant No. ECS90-06921.

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